LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

Calculus III

Exam I

Fall 2016

Duration: 75 minutes. No calculators allowed.

Name: Solution **ID:** ______

Circle the name of your instructor: **Dr. Hamdan** **Dr. Issa** **Dr. Touma**

Question Number	Grade
1. 21%	
2. 16%	
3. 28%	
4. 8%	
5. 12%	
6. 15%	
TOTAL	

Problem 1: (21%) Evaluate the following integrals

(a) $\int \frac{dx}{\sqrt{6x - x^2}}$

$$x^2 - 6x + 9 - 9$$
$$(x-3)^2 - 9$$

$$= \int \frac{dx}{\sqrt{9 - (x-3)^2}}$$

$$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$$

(b) $\int \frac{e^{3x}}{e^{2x} + 4}\, dx$

$u = e^x$

$$\int \frac{u^2}{u^2+4}\, du = \int \frac{u^2+4-4}{u^2+4} = \int du - \int \frac{4}{u^2+4}\, du.$$

$$= u - 4 \cdot \frac{1}{2} \tan^{-1}(u/2) = e^x - 2\tan^{-1}(e^x/2) + C$$

(c) $\int \frac{\sinh x}{\sqrt{\cosh x}}\, dx$ $= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du$

$u = \cosh x$

$$= 2\sqrt{u}$$

$$= 2\sqrt{\cosh x} + C.$$

Problem 2: (16%) Evaluate the following improper integrals

(a) $\int_0^2 \frac{1}{x^3+x}\,dx \quad = \int_0^2 \frac{dx}{x(x^2+1)}$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow \begin{array}{l} Ax^2 + A + Bx^2 + Cx = 1 \\ (A+B)x^2 + Cx + A = 1 \end{array}$$

$$C = 0$$
$$A = -B$$
$$A = -1$$
$$B = 1$$

$$= \int -\frac{1}{x} + \frac{x}{x^2+1}$$

$$= -\ln x + \frac{1}{2}\ln(x^2+1) = \ln\left(\frac{\sqrt{x^2+1}}{x}\right)$$

$$\therefore \text{Int} = \lim_{t\to 0}\int_t^2 \cdots = \lim \ln\frac{\sqrt{5}}{2} - \ln(\infty) \rightarrow \text{Divg}$$

(b) $\int_{-\infty}^2 \frac{1}{x^2+2x+10}\,dx$

$$x^2+2x+10 = x^2+2x+1+9 = (x+1)^2+9.$$

$$\int_{-\infty}^2 \frac{dx}{(x+1)^2+9} = \lim_{t\to-\infty} \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right)\Big|_t^2$$

$$= \frac{1}{3}\left(\tan^{-1}(1) - \tan^{-1}(-\infty)\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) \rightarrow \frac{\pi}{4}$$

Problem 3: (28%) For the following integrals, state whether they are convergent or divergent, and give your reasons.

(a) $\int_1^\infty \frac{x^3}{\ln x + x^4}\,dx$

$$\frac{x^3}{x^4+\ln x} \approx \frac{1}{x}$$

$\therefore \int_1^\infty \frac{dx}{x}$ div. $\Rightarrow$ Div. by LCT.

(b) $\int_0^\infty \frac{x^{10}}{e^{\sqrt{x}}}\,dx$

$$= \int_0^1 + \int_1^\infty$$

bounded

$e^{\sqrt{x}} > x^{12}$ for large x

$$\therefore \frac{1}{e^{\sqrt{x}}} < \frac{1}{x^{12}}$$

$$\Rightarrow \frac{x^{10}}{e^{\sqrt{x}}} < \frac{1}{x^2} \Rightarrow$$

$$\int_1^\infty \frac{x^{10}}{e^{\sqrt{x}}}\,dx < \int_1^\infty \frac{1}{x^2} \Rightarrow \text{Conv. by DCT.}$$

p-int. $p=2>1 \Rightarrow$ conv.

$\therefore$ The whole int. conv.

(c) $\int_2^\infty \frac{x^2+3x+2}{(\sqrt{x}-1)^3\sqrt{x^3-1}}\,dx$ $\;\tilde{\sim}\; \int \frac{x^2}{x^{1.5} \cdot x^{1.5}}$

$= \int \frac{x^2}{x^3} \overset{LCT}{\tilde{\sim}} \int \frac{1}{x} \Rightarrow$ div. by LCT.

(d) $\int_2^\infty \frac{(\ln x)^{10}}{x\sqrt{x}}\,dx$

$(\ln x)^{10} < x^{0.1}$

$\therefore \frac{(\ln x)^{10}}{x^{1.5}} < \frac{1}{x^{1.5-0.1}} = \frac{1}{x^{1.4}}$

$\therefore$ Our int. $< \int \frac{dx}{x^{1.4}}$ conv. p-int $p>1$.

$\therefore$ our int $<$ conv. int. $\Rightarrow$ Conv. by DCT.

Problem 4: (8%) Suppose that $f(x)$ is non-negative, continuous over $[a, \infty)$ and that $\int_a^\infty f(x)\,dx$ converges. What can you say about the convergence or divergence of $\int_a^\infty \frac{1}{1+f(x)}\,dx$? Justify.

Must have $f(x) \longrightarrow 0$ (because $\int_a^\infty f(x)\,dx$ conv. $\Rightarrow$

$$\frac{1}{1+f(x)} \longrightarrow 1 \neq 0 \therefore \quad \int_a^\infty \frac{1}{1+f(x)} \approx \int_a^\infty 1\,dx = \infty$$

$\Rightarrow$ div.

Problem 5: (12%) For which $a \in \mathbb{R}$ does the following improper integral converge? Justify your answer.

$$\int_1^\infty \left(\frac{1}{x+2} - \frac{ax}{x^2+1} \right) dx$$

$$\frac{1}{x+2} - \frac{ax}{x^2+1} = \frac{x^2+1 - ax(x+2)}{(x+2)(x^2+1)}$$

$$\approx \frac{(1-a)x^2 - 2ax + 1}{x^3}$$

In order to conv. must have $a=1$,

because then, our integrand $\approx \frac{-2ax}{x^3} \approx \frac{1}{x^2}$.

$\therefore \boxed{a=1}$

4 'if recall $(a-1)x$ etc.'

Problem 6: (15%) Determine if the following sequences converge or diverge. Justify your answers.

(a) $a_n = \dfrac{n^2 \sin\left((2n-1)\frac{\pi}{2}\right)}{n+1}$ $= \dfrac{n^2 (-1)^n}{n+1} \approx \dfrac{n}{1} \longrightarrow \infty$

~~conv to zero~~

Diverges

(b) $a_n = \left(1 - \dfrac{2}{n}\right)^n \dfrac{1}{\sqrt[n]{n^3}} \longrightarrow e^{-2}.$

$\left(1 - \dfrac{2}{n}\right)^n \to e^{-2}$

$\dfrac{1}{\sqrt[n]{n^3}} \to 1$

(c) $a_n = e^{n+1}\sin\left(e^{-n}\right)$

$e^{-n} \to 0$

$\sin(e^{-n}) \to 0$

$= e \cdot e^n \sin e^{-n}$

$= e \cdot \left(\dfrac{\sin(e^{-n})}{e^{-n}}\right)$

$e^{-n} \to 0$

$\dfrac{\sin(e^{-n})}{e^{-n}} \to 1$ (of the form $\dfrac{\sin x}{x} \xrightarrow[x \to 0]{} 1$)

$= e(1) \longrightarrow$ conv. to $\boxed{e}$